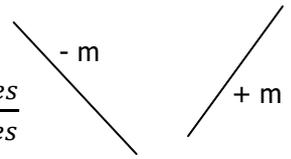


GCSE Higher Maths Key Notes

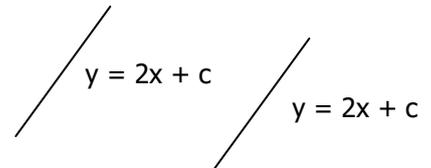
- ❖ Equations of straight lines **are always** in this form: $y = mx + c$

$m =$ gradient (steepness of the line) $m = \frac{\text{difference in } y\text{-values}}{\text{difference in } x\text{-values}}$

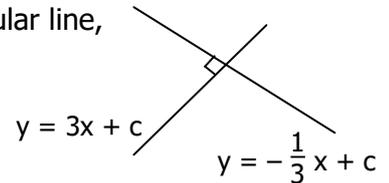


$c =$ where it crosses the y – axis so $y = mx + c$ will cross the y -axis at **(0, c)**
use an actual point (x₁, y₁) on the line to find 'c'

- ❖ For parallel lines, the gradient is **obviously the same**



- ❖ For perpendicular lines, **the gradients will multiply to give you – 1**
 so if you have one gradient, to find the gradient of the perpendicular line,
 just **change the sign and take the reciprocal**



- ❖ What does a box plot show? Mainly the **inter-quartile range and the median**

Therefore, when comparing box plots, describe the I.Q.R and medians only.

A greater median means greater results, on average.

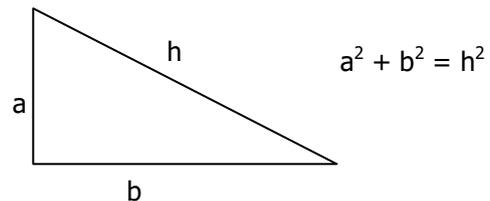
A greater I.Q.R (middle 50% of the data) means greater spread.

Don't forget to apply it to context so for example, the question could be on science results.

- ❖ When you see a right-angled triangle, it can only mean **two things**:

1) **Pythagoras theorem:** When two lengths are given and you trying to find a third.

*Note: 'h' is the hypotenuse. It is the longest side of the triangle and therefore, **it's always the same!***

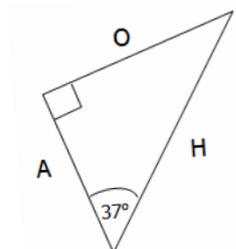


Also, you can **find the length of a line (y = mx + c)** using Pythagoras theorem

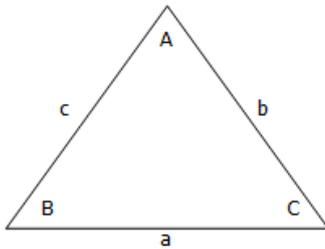
2) **Trig Ratios (SOH CAH TOA)**

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \quad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \quad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$

Remember 'h' is always the same but '**o' and 'a' can vary depending on the angle in question** (see diagram). 'o' is opposite the angle and 'a' is next to the angle, in question.



❖ Any triangle: **Cosine & Sine Rules**



You don't have to label it like this, I just happened to do it this way.

You label it based on what's given to you in the question and what formula (and version) you are going to use.

And make sure the letters correspond with the angles/sides and use capital letters to define the angles!

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$a^2 = b^2 + c^2 - 2bc(\cos A) \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

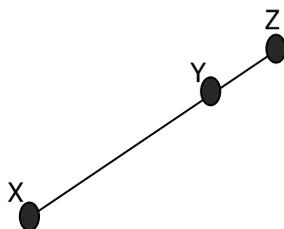
You don't have to remember these formulas but you need to know when to use them. Below is a table of when (and how) to use them.

| What is given to you? | Sine or Cosine? | What version should you use? | Why should you use this? |
|--|-----------------|--|--|
| Two angles + any side | Sine | $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ | Because you are going to find a side |
| Two sides + an angle NOT enclosed by those two sides | Sine | $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ | Because you are going to find an angle |
| Two sides + the enclosed angle | Cosine | $a^2 = b^2 + c^2 - 2bc(\cos A)$ | Because you are going to find a side |
| All three sides given and no angles | Cosine | $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ | Because you are going to find an angle |

❖ Proof questions: **Vectors**

1) Prove that two vectors are parallel: **First, find these two vectors. Then, find the common vector between the two.** You may have to factorize the vectors. Remember vectors behave the same way as algebra so look for the HCF when factorizing!

2) Show the points X, Y and Z lie on the straight line.



1) Find the vector of the whole line so in this case, it would be \vec{XZ}

2) Then choose the smaller vector on the line. In this case, it would be \vec{YZ}

3) Show that the smaller vector divides into the larger vector i.e. \vec{YZ} goes into \vec{XZ}

The way to do this is to show that $\vec{XZ} = a\vec{YZ}$ for some number 'a'

❖ Similar Shapes: **Lengths, Areas/Surface Area, Volume**

Remember these ratios:

$$\left(\frac{\text{side length of bigger object}}{\text{side length of smaller object}} \right)^2 = \frac{\text{Area or Surface area of bigger object}}{\text{Area or Surface Area of smaller object}}$$

$$\left(\frac{\text{side length of bigger object}}{\text{side length of smaller object}} \right)^3 = \frac{\text{volume of bigger object}}{\text{volume of smaller object}}$$

just plug in what you know and rearrange it to find what you're looking for!

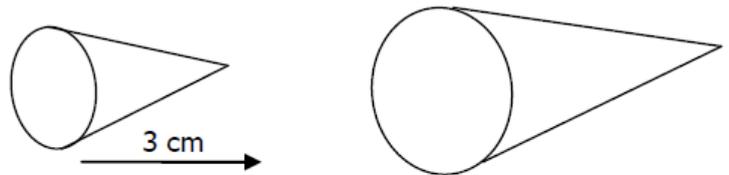
Or, if the above is difficult to remember, you could think of it this way...

General rule: Given that the ratio between the lengths of two mathematically similar shapes are **a : b**, then:

The ratio between their areas or surface areas is $a^2 : b^2$

The ratio between their volumes is $a^3 : b^3$

Exam-style Question: Two cones are mathematically similar. The height of the smaller cone is 3 cm. The volume of the smaller cone is 4π and the volume of the larger cone is 108π . What is the height of the larger cone?



Answer: The ratio of their volumes are... $4\pi : 108\pi$ \longleftrightarrow $1 : 27$
 The ratio of their heights are... $\sqrt[3]{1} : \sqrt[3]{27}$ \longleftrightarrow $1 : 3$
 So the height of the larger cone is just... $3 \times 3 = 9\text{cm}$

This is like the scale factor of the heights.

❖ **Solving Quadratic Equations** ($ax^2 + bx + c = 0$) when $a = 1$

1) First try to solve it **by inspection**. This means set it out into two brackets:

$$(x \quad)(x \quad) = 0 \quad \text{Find two values that multiply to give 'c' and add to give 'b'}$$

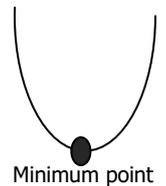
If you can't do this, then...

2) **Completing the square:**

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \quad \text{just sub-in 'b' and 'c' and simplify.}$$

When you've simplified it, the final form would just look like this: $(x + s)^2 + t$ where 's' and 't' are just numbers!

Note: This is the **only way you can find a minimum point** of a quadratic equation. You have to complete the square on it and then the minimum is when this part of the equation $(x + s)^2 = 0$. The x-value would just be '-s'. The y value would therefore be t. So minimum point is $(-s, t)$



3) If you don't fancy using the above two methods, then you can always use the quadratic equation (given in the exam):

The solutions of $ax^2 + bx + c = 0$
where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

❖ **Solving Quadratic Equations** ($ax^2 + bx + c = 0$) when $a > 1$

1) First see if 'a' can divide into 'b' and 'c'. If so, divide everything by 'a' and you will go back to the case when $a = 1$ i.e. $x^2 + bx + c$ where b and c are integers (see above).

If you cannot divide by 'a', read on...

2) **By inspection:**

If 'a' is prime, it **must be** in this form $(ax \quad)(x \quad) = 0$

And proceed as before – two numbers that multiply to give 'c' but add to give 'b'

If 'a' is not prime, it will take slightly longer and you have to try all the different combinations (trial and error). Or, you could use the grouping method.

If you cannot put it into two brackets, then **you would have to use the quadratic formula (see above)**.

❖ **Simplifying fractions** which have quadratic expressions ($ax^2 + bx + c$) as the numerator and denominator:

1) See if you can use the difference of two squares formula i.e.

$a^2 - b^2 = (a + b)(a - b)$ to put the expression into two brackets. See below:

$$\frac{x^2 - 16}{x^2 - 4x} = \frac{(x + 4)(x - 4)}{x(x - 4)} = \frac{(x + 4)}{x}$$

2) Always factorise the quadratic (with $a = 1$) first. Because this will give you a big clue on how to factorise the case $a > 1$. Look at the example below:

$$\frac{3x^2 - 7x - 6}{x^2 + 2x - 15} = \frac{3x^2 - 7x - 6}{(x + 5)(x - 3)} = \frac{(3x + 2)(x - 3)}{(x + 5)(x - 3)} = \frac{(3x + 2)}{(x + 5)}$$

After factorising the bottom, you will know that the top will contain either a $(x + 5)$ or a $(x - 3)$ because one of the brackets will eventually cancel.

If you look at the top, 'a' is prime so it must be in the form $(3x + \quad)(x - \quad)$. There are only 2 possibilities that multiply to give $c (= 6)$. Don't worry about the signs for now. The possibilities are either 1 and 6, or 2 and 3. But if you look at the bottom, we found it contains either a $(x + 5)$ or a $(x - 3)$. It cannot be $(x + 5)$ on the top as 5 is not in any of the pairs. So it must be $(x - 3)$ and the other bracket is therefore, $(3x + 2)$. Then, lastly, cross-out the $(x - 3)$ on the top and bottom.

❖ **Functions of Graphs $f(x)$:**

Don't be put off by $f(x)$, it just stands for 'a function of x '. You are very familiar with these by now. Functions of x are any one of these:

$$f(x) = x^2 + 2$$

$$f(x) = 3x + \frac{2}{3}$$

$$f(x) = -\frac{1}{x}$$

These are all functions because we can sub-in values of x and it will generate a new value i.e. in the first one, when we sub in $x = 2$,

$f(2) = (2)^2 + 2 = 6$. All of these graphs can obviously be plotted.

❖ **Graph Transformations:**

Suppose we have any function $y = f(x)$. Then:

1) $y = f(x) + a$ is a transformation of the $y = f(x)$ graph by a translation of $\begin{pmatrix} 0 \\ a \end{pmatrix}$ i.e. it moves 'a' units up the y axis.

2) $y = f(x + a)$ is a transformation of the $y = f(x)$ graph by a translation of $\begin{pmatrix} -a \\ 0 \end{pmatrix}$ i.e. it moves 'a' units left along the x axis.

3) $y = af(x)$ is a transformation of the $y = f(x)$ graph by a stretch of scale factor 'a' up the y axis. In simple terms, the x-values stay the same whereas the y-values are multiplied by a!

4) $y = f(ax)$ is a transformation of the $y = f(x)$ graph by a stretch of scale factor $\frac{1}{a}$ along the x axis. In simple terms, the y values stay the same whereas the x-values are multiplied by $\frac{1}{a}$ (kind of like the opposite to the previous rule)

❖ **Finding an equation from a graph:**

1) Sub in a point which is on one of the axis. Why? Because **we will eliminate one of the variables.**

2) Sub-in the other point to find the full equation.

Look at this exam question to understand what I mean:

Mr Patel has a car.

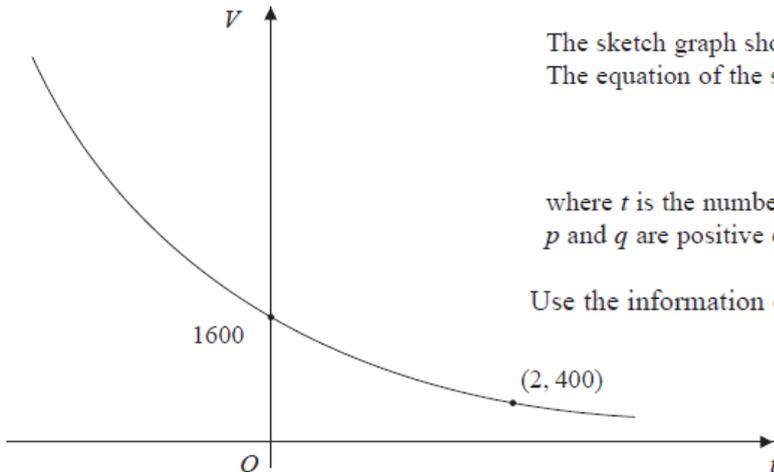
The value of the car on January 1st 2000 was £1600
The value of the car on January 1st 2002 was £400

The sketch graph shows how the value, £ V , of the car changes with time.
The equation of the sketch graph is

$$V = pq^t$$

where t is the number of years after January 1st 2000.
 p and q are positive constants.

Use the information on the graph to find the value of p and the value of q .



This is very similar to a x-y graph but they've replaced y with V and x with t.
To solve this, first we would sub-in the point on the y-axis ($t = 0, V = 1600$). This gives us:

$$1600 = pq^0 \quad \rightarrow \quad 1600 = p \quad (\text{we have found } p \text{ quite easily})$$

Next, we just sub-in the other point ($t = 2, V = 400$) to find q ...

$$400 = 1600(q)^2 \quad \rightarrow \quad \frac{1}{4} = q^2 \quad \rightarrow \quad \frac{1}{2} = q$$

Note: q is positive. That's why I ignored the solution $q = -\frac{1}{2}$

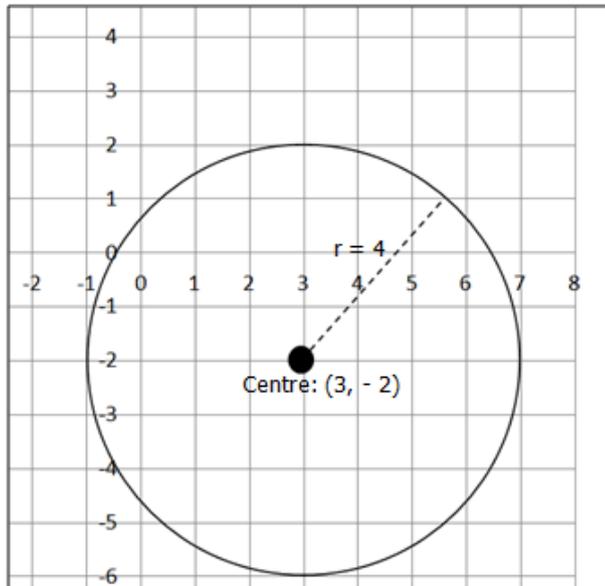
So our full equation is just $V = 1600(\frac{1}{2})^t$

❖ **Equation of a circle:**

$$(x - a)^2 + (y - b)^2 = r^2$$

where the point (a, b) is the centre of the circle and r is the radius.

$$(x - 3)^2 + (y + 2)^2 = 16$$



❖ **Proportionality** – When you set up your equation, don't forget that there is a constant, k, in there.

If the question says, inversely proportional, then it will look like $\frac{k}{\dots}$

If the question says, directly proportion, then it will look like $k \times \dots$

Remember, you **have to find this 'k'** to complete the equation. They should give you a point (or set of values) so you can find 'k'. Then, lastly put 'k' into your original equation.

❖ **Surds** – Remember surds is just a fancy name for a root of a number i.e. $\sqrt{3}$
 These roots are usually irrational as well (never ending, non repeating decimals)
 i.e. $\sqrt{3} = 1.73205080\dots$ That's why it's best to leave it in 'surd' form instead of taking the whole value and rounding etc. because you will lose accuracy!

Remember when you multiply a surd by itself, you go back to the original number
 i.e. $\sqrt{a}\sqrt{a} = \sqrt{(a \times a)} = \sqrt{a^2} = a$

❖ **Rationalising the denominator** – This means making the denominator rational if it is currently irrational. Therefore, the denominator is generally a surd or a bracket containing a surd(s). To rationalise the denominator you just multiply top and bottom by the surd, if it's just a single surd on the bottom. Or, if it's a bracket of

surd(s), you multiply by the same bracket top and bottom but flip the sign in the bracket. Take a look at the next two examples:

Example: Rationalise $\frac{5}{\sqrt{7}}$

$$\frac{5}{\sqrt{7}} = \frac{5 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{5\sqrt{7}}{7} = \frac{5}{7}\sqrt{7}$$

Example: Rationalise $\frac{2}{6 + 3\sqrt{5}}$

$$\begin{aligned} \frac{2}{6 + 3\sqrt{5}} &= \frac{2(6 - 3\sqrt{5})}{(6 + 3\sqrt{5})(6 - 3\sqrt{5})} \\ &= \frac{12 - 6\sqrt{5}}{(6)^2 - (3\sqrt{5})^2} = \frac{12 - 6\sqrt{5}}{(36) - (9 \times 5)} \\ &= \frac{12 - 6\sqrt{5}}{-9} = -\frac{3(4 - 2\sqrt{5})}{9} \\ &= -\frac{(4 - 2\sqrt{5})}{3} = \frac{-4 + 2\sqrt{5}}{3} \end{aligned}$$

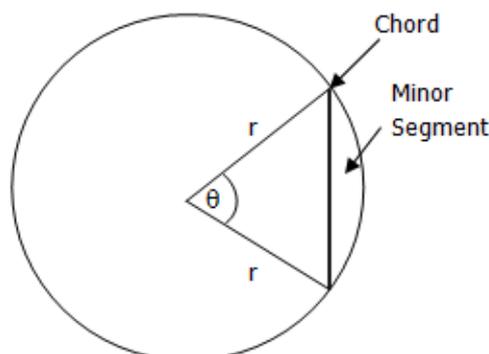
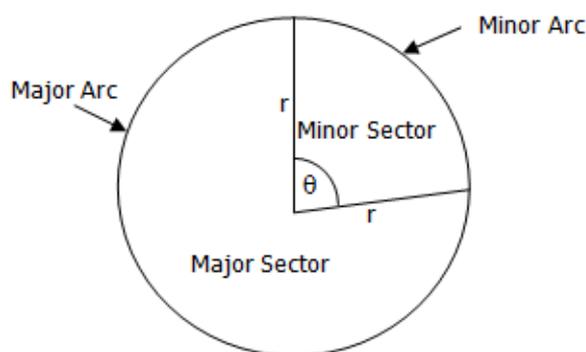
Here we change the sign of the surd and multiply it at the top and the bottom, to rationalise the denominator.

We will always get a difference of two squares here!

3 is a common factor so we factorize the numerator and of course this cancels with the denominator (9).

❖ Arc Length & Area of Minor Sector

Area



$$\text{Area of minor sector} = \frac{\theta}{360} \times \pi r^2$$

$$\text{Length of minor arc} = \frac{\theta}{360} \times \pi d$$

Remember, r stands for radius and d stands for diameter. These formulas are similar to the area and circumference of a circle. The only difference is we multiply it by the proportion $\frac{\theta}{360}$ at the beginning.

To find the area of the minor segment:

- 1) Find the area of the entire minor sector using the formula above.
- 2) Find the area of the triangle: you may have to use trigonometry to do this (more on this later).
- 3) Subtract (2) from (1) to get the area of the segment. Can you see why you have to do this?

❖ **Solving Inequalities:** These behave exactly the same way as normal equations but instead you have an inequality sign ($<$, \leq , $>$, \geq) where the equals ($=$) is. Solve it as normal but **if ever you have to multiply or divide by a negative number, change the direction of the inequality sign** so ' $<$ ' becomes ' $>$ ' or ' \geq ' becomes ' \leq '.

❖ **Making a 'letter' the subject of the formula: Remember factorizing makes a letter become 'singular'.** For instance, in the following example, you may want to make 'p' the subject of the formula:

$$4p - pq = 18$$

Clearly you cannot simplify the terms on the L.H.S. That's why you just factor out the p so it becomes singular:

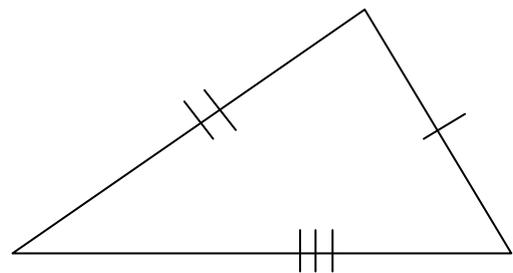
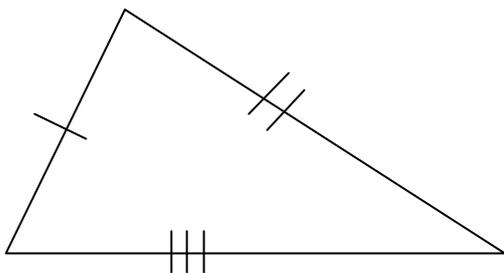
$$p(4 - q) = 18. \quad \text{Then move the } (4 - q) \text{ to the R.H.S and you're left with p.}$$

Congruent Triangles (higher tier only)

To prove that two triangles are congruent, you have to make sure their sides/angles match up. Here are the 4 rules to show that two triangles are congruent:

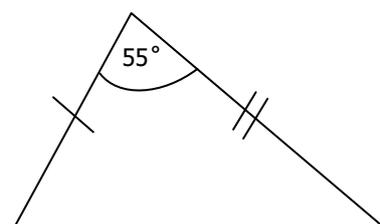
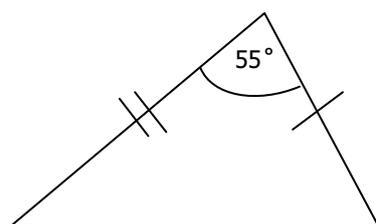
1. All three sides are the same (SSS)

The SSS (side, side, side) rule to show that the sides of the first triangle are the same as the second triangle.



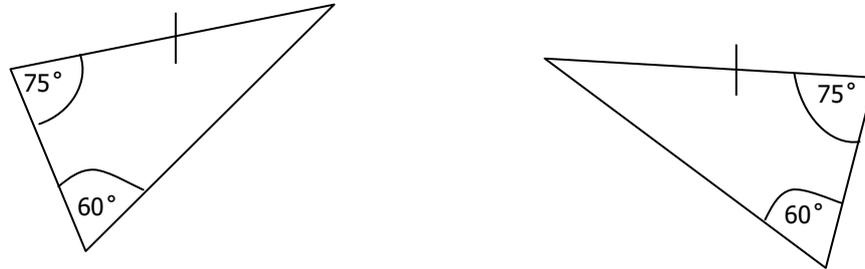
2. Two sides and the included angle (SAS)

The SAS (side, angle, side) rule to show the two sides and the enclosed angle of the first triangle is the same as the second triangle.



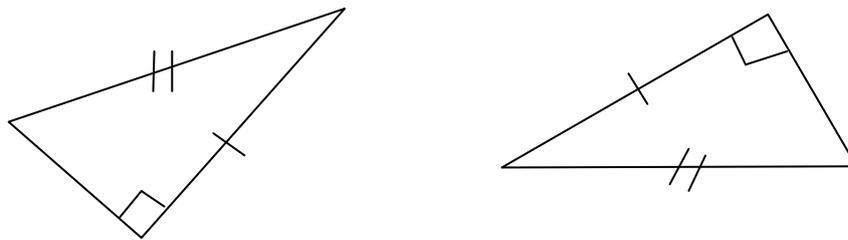
3. Two angles and one side are the same

The AAS (angle, angle, side) rule to show two angles in the first triangle are the same as the two angles in the second triangle, including a similarly located side.



4. Two sides in a right-angled triangle (RHS)

The RHS (right-angled, hypotenuse, side) rule to show that the hypotenuse and one other side in the first triangle is the same as the second triangle.

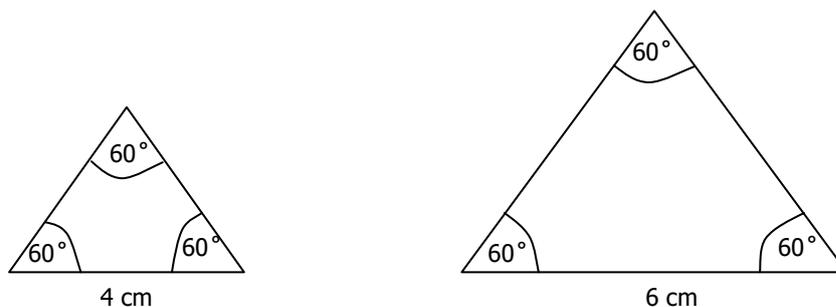


Tip:

Think about what's given to you in the diagram first. Is there any common angles or sides? Then, think about what rule is most appropriate to use.

Note:

You notice that there isn't an angle, angle, angle (AAA) rule where all 3 angles are the same. This is because it doesn't prove a triangle (or any shape) is congruent. This just proves that they're mathematically similar. Consider the equilateral triangles below:



*Not drawn to scale.
It's just to explain
the point.*

The smaller triangle has a side length of 4 cm and the larger triangle has a side length of 6 cm. Obviously, the angles are the same (60°). This means proving that all the angles are the same does not necessarily mean it's congruent. They are only similar. You have to prove at least one of the sides are the same as well to show that it's congruent. Keep this in mind.

- ❖ **Inequalities (Regions):** Firstly, turn the inequalities into straight line equations. This is easy – just replace the inequality signs with an equals sign. Draw these straight lines (you should be able to do this if you're a higher-tier student!). Once, you've plotted them, you can think of them as the **boundary lines**. Then, consider the original inequalities. Remember, the region is **NOT always** the area which is enclosed by all the straight lines. You have check on what side of the line, each inequality is satisfied. Maybe take some points (x_1, y_1) on the graph to see if the inequality is satisfied. Lastly, if the inequality signs are ' \leq ' or ' \geq ', then **it can touch the boundary lines**. Think about why?
- ❖ **Equations with algebraic fractions in them:** The key in any equation (and not just algebraic ones) is to get rid of the denominators of any fraction. To do this, you just multiply throughout the whole equation by these denominators. Then, you'll get a 'straight line' (terms in a straight line) equation. Once you have this you just solve it as normal...

Question: Solve the equation

$$\frac{4}{2x+1} - \frac{1}{3x-1} = 5$$

First you multiply everything by $(2x+1)$ and then everything by $(3x-1)$. And now we have a straight line equation.

Answer: $4(3x-1) - (2x+1) = 5(3x-1)(2x+1)$

$$10x - 5 = 5(6x^2 + x - 1)$$

$$5(2x - 1) = 5(6x^2 + x - 1)$$

$$2x - 1 = 6x^2 + x - 1$$

Solve as normal; expand out the brackets, rearrange and simplify. You can sense this is going to be a quadratic equation!

$$6x^2 - x = 0$$

$$x(6x - 1) = 0$$

$$x = 0 \text{ or } x = \frac{1}{6}$$

You should always factorise a quadratic equation because it gives you more than 1 solution. At $6x^2 - x = 0$, you could have moved the 'x' to the RHS and divided by x to give you $6x = 1$ and so $x = \frac{1}{6}$ but do you see how we would have forgotten the $x = 0$ solution?? This is a valid solution too! Check by plugging it in to the original equation! So if you forget this, you could potentially lose a mark.